On covers of graphs by Cayley graphs

Agelos Georgakopoulos*

Mathematics Institute
University of Warwick
CV4 7AL, UK

Abstract

We prove that every vertex transitive, planar, 1-ended, graph covers every graph whose balls of radius r are isomorphic to the ball of radius r in G for a sufficiently large r. We ask whether this is a general property of finitely presented Cayley graphs, as well as further related questions.

1 Introduction

We will say that a graph H is *r-locally-G* if every ball of radius r in H is isomorphic to the ball of radius r in G. The following problem arose from a discussion with Itai Benjamini, and also appears in [5].

Problem 1.1. *Does every finitely presented Cayley graph G admit an $r \in \mathbb{N}$ such that G covers every r-locally-G graph?*

The condition of being finitely presented is important here: for example, no such r exists for the standard Cayley graph of the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2$.

Benjamini & Ellis [4] show that $r = 2$ suffices for the square grid $\mathbb{Z}^2$, while $r = 3$ suffices for the d-dimensional lattice (i.e. the standard Cayley graph of $\mathbb{Z}^d$ for any $d \geq 3$.

The main result of this paper is

Theorem 1.1. *Let G be a vertex transitive planar 1-ended graph. Then there is $r \in \mathbb{N}$ such that G covers every r-locally-G graph (normally).*

Here, we say that a cover $c : V(G) \to V(H)$ is *normal*, if for every $v, w \in V(G)$ such that $c(v) = c(w)$, there is an automorphism α of G such that $\alpha(v) = \alpha(w)$ and $c \circ \alpha = c$. If $c : V(G) \to V(H)$ is a normal cover, then H is a quotient of G by a subgroup of $Aut(G)$, namely the group of 'covering transformations'; see [4, Lemma 16] for a proof and more details. Normality of the covers was important in [4], as it allows one to reduce enumeration problems for graphs covered by lattices to counting certain subgroups of $Aut(G)$.

A natural approach for proving Theorem 1.1 is by glueing 2-cells to the r-locally-G graph H along cycles that map to face-boundaries of G via local isomorphisms to obtain a surface S_H, and consider the universal covering map

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$c : \mathbb{R}^2 \to S_H$. Then $c^{-1}[H]$ is a 1-ended graph G' embedded in $\mathbb{R}^2$ which is also r-locally-G, and if we could show that G' is isomorphic to G we would be done. The latter statement however turns out to be as hard as Theorem 1.1 itself, and in fact we will obtain it as a byproduct of our proof[1].

Let us call an infinite group *weakly residually finite*, if all its Cayley graphs G have the following property: for every $r \in \mathbb{N}$, there is a finite graph H which is r-locally-G. It is not hard to prove that every residually finite group is weakly residually finite. Indeed, given a Cayley graph G of a residually finite group Γ and some r, we can find a homomorphism h from Γ to a finite group Δ which is injective on the ball of radius r around the origin of G. Then the Cayley graph of Δ with respect to the generating set $h[S]$, where S is the generating set of G, is indeed r-locally-G. Is the converse statement also true, that is,

Problem 1.2. *Is every weakly residually finite group residually finite?*

If this is true it would yield an alternative definition of residually finite groups. If not, studying the relationship between weakly residually finite and sofic groups might be interesting. Similar questions can be asked using graphs covered by G rather than r-locally-G graphs.

Benjamini & Ellis [3] consider the uniform probability distribution on the r-locally-G graphs with n vertices for $G = \mathbb{Z}^n$, and study properties of this random graphs as n grows. They do so by exploiting normal covers in order to reduce the enumeration of such graphs to the enumeration of certain subgroups of $Aut(G)$, which had previously been studied. Theorem 1.1 paves the way for the study of the uniformly random r-locally-G graph H_n on n vertices, with G being e.g. a regular hyperbolic tessellation. The genus of H_n can easily be seen to be linear in n in our case (while it was always 1 in [3] for $G = \mathbb{Z}^2$). Glueing metric 2-cells to H_n as described above we obtain a random closed Riemannian surface. I hope that this topic will be pursued in future work.

Our r in Theorem 1.1 can be arbitrarily large. It is not clear from our proof whether there is an upper bound depending on the maximum co-degree (i.e. length of a face) of G only, or it also depends e.g. on the vertex degree. The results of [7] might be helpful for answering this question.

Tessera and De La Salle (private communication) recently announced a positive answer to Problem 1.1 under the condition that $Aut(G)$ is discrete, and a counterexample showing that this condition is necessary.

2 Preliminaries

A graph G is *(vertex) transitive*, if for every two vertices v, w there is an automorphism of G mapping v to w. The group of automorphisms of G is denoted by $Aut(G)$. We say that $Aut(G)$ is *discrete*, if the stabiliser of each vertex is finite.

A *cover* from a graph G to a graph H is a map $c : V(G) \to V(H)$ such that the restriction of c to the neighbourhood of any vertex of G is a bijection.

[1] I would like to thank Bojan Mohar for suggesting this approach.

2.1 Planar graphs

A *plane graph* is a graph G endowed with a fixed embedding in the plane $\mathbb{R}^2$; more formally, G is a plane graph if $V(G) \subset \mathbb{R}^2$ and each edge $e \in E(G)$ is an arc between its two vertices that does not meet any other vertices or edges. A graph is *planar* if it admits an embedding in $\mathbb{R}^2$. Note that a given planar graph can be isomorphic (in the graph-therotic sense) to various plane graphs that cannot necessarily be mapped onto each other via a homeomorphism of $\mathbb{R}^2$.

A *face* of a planar embedding is a component of the complement of its image, that is, a maximal connected subset of the plane to which no vertex or edge is mapped. The *boundary* of a face is the set of edges in its closure.

Lemma 2.1 ([9])**.** *Let G be a vertex transitive plane 1-ended graph. Then every face-boundary of G contains only finitely many edges.*

This means that every face-boundary is a cycle of G in our case.

Given a planar embedding of a graph G, we define a *facial path* to be a path of G contained in the boundary of a face. We define a *facial walk* similarly.

The following is a classical result, proved by Whitney [11, Theorem 11] for finite graphs. It extends to infinite ones by compactness; see [8].

Theorem 2.2. *Let G be a 3-connected graph embedded in the sphere. Then every automorphism of G maps each facial path to a facial path.* □

The *connectivity* of a graph is the cardinality of a smallest vertex set whose deletion disconnects the graph. A graph is 3-*connected* if its connectivity is at least 3. The next result is due to Babai and Watkins [2], see also [1, Lemma 2.4].

Lemma 2.3. [2, Theorem 1] *Let G be a locally finite connected transitive graph that has precisely one end. Let d be the degree of any of its vertices. Then the connectivity of G is at least $3(d+1)/4$.* □

We deduce from Lemma 2.3 and Theorem 2.2 that for every 1-ended transitive planar graph, face-boundaries depend only on the graph and not on any embedding we might choose.

2.2 Graphs that are locally planar

Given a graph H that is r-locally-G, where G is planar, we would like to be able to talk about 'face-boundaries' of H, although H is not necessarily planar itself. This can be done by using the notion of a *peripheral cycle*. Recall that a cycle $C = v_0, v_1 \ldots, v_k = v_0$ of a graph H is *induced*, if G contains no edge from v_i to v_j for $|i-j| > 1(mod\ k)$. A cycle C is *peripheral* if it is both induced and non-separating. If G is a connected plane graph then each peripheral cycle bounds a face of G. If G is also 3-connected, then every face-boundary is peripheral.

A *flag* of a plane graph G is a triple $\{u, e, F\}$, consisting of a vertex u, an edge e, and a face-boundary F, such that $u \in e \in F$. We denote by $\mathcal{F} = \mathcal{F}(G)$ the set of all flags of G.

Note that by Whitney's theorem, every automorphism of G can be naturally extended to the flags of G.

For a vertex o of G, the *ball* $B_i(o; G)$ of radius i —also denoted by $B_i(o)$ if G is fixed— is the subgraph of G induced by the vertices at graph-distance at

most i from o. As we are dealing with planar graphs, it is more convenient to consider the following variant:

Definition 2.4. *We let $D_k(o;G)$ denote $B_j(o;G)$ for the smallest $j \in \mathbb{N} \cup \{\infty\}$ such that $B_j(o;G)$ contains every vertex $v \in V(G)$ for which there is a sequence of peripheral cycles $C_1, \ldots, C_k$ with $o \in V(C_1), v \in V(C_k)$, and $C_i \cap C_{i+1} \neq \emptyset$ for every relevant i.*

Note that if G is planar, then every peripheral cycle bounds a face, and so j is finite. In this case $D_i(o;G)$ is a ball of G large enough to contain the ball of radius i of the dual of G, but the definition also makes sense for non-planar graphs.

Lemma 2.5. *Let G be a vertex transitive plane 1-ended graph, and $o \in V(G)$. Then the face-boundaries of G containing o coincide with the peripheral cycles of $D_2(o)$.*

Proof. Let F be a face-boundary incident with o. By Lemma 2.1, F is a finite cycle $v_1(=o)v_2 \ldots v_k = v_1$. Clearly, F is induced in $D_2(o)$; we will show it is non-separating. It is not hard to prove (see e.g. [6, Lemma 1.1]) that $P_i := D_1(v_i) \backslash F$ is a path for every $1 \leq i \leq k$. It follows that $F' := \bigcup_{1 \leq i \leq k} P_i$ is connected (in fact, it is a cycle). Moreover, F' separates F from $G \backslash F$ by construction. To show that F does not separate $D_2(o)$, notice that if Q is a path with both its endvertices outside F with $Q \cap F \neq \emptyset$, then Q meets F' and can be shortcut into a path with the same endvertices avoiding F. Thus F is peripheral in $D_2(o)$.

Conversely, let F be a peripheral cycle of $D_2(o)$ containing o. Then F is a face-boundary in any embedding of $D_2(o)$ (or G) as remarked above. □

This lemma justifies the following definition, which allows us to retain our intuition of faces in an r-locally-G graph which is not necessarily planar.

Definition 2.6. *Let G be a vertex transitive plane 1-ended graph, and H a graph which is r-locally-G for some $r \geq 2$. We define a* face-boundary *of H to be any peripheral cycle of $D_2(v;H)$ incident with v for any $v \in V(H)$. We extend the definition of a flag, and that of a facial walk, to such graphs H using this notion of face-boundary.*

2.3 Automorphisms, flags, and fundamental domains

Theorem 1.1 is easier to prove when the face-boundaries incident with one (and hence each) vertex have distinct sizes. Complications arise when this is not the case, especially when the automorphism group of G has non-trivial vertex stabilizers. In order to deal with these complications, we adapt the standard notion of a fundamental domain to our planar setup as follows. We fix a vertex $o \in V(G)$, and define a *fundamental domain* of G to be a connected sequence of flags of o containing exactly one flag from each orbit of $Aut(G)$. Here, we say that a sequence $f_1, \ldots, f_k$ of flags of o is *connected*, if f_i is incident with f_{i+1} for every $1 \leq i < k$, and we say that $\{o, e, F\}$ is *incident* to $\{o, e', F'\}$ if either $e = e'$ or $F = F'$. For $i \in \mathbb{N}$, we define an *i-fundamental domain* of G similarly except that we replace $Aut(G)$ by $Aut(D_i(o))$.

3 Proof of Theorem 1.1

Lemma 3.1. *There is $n \in \mathbb{N}$ such that every n-fundamental domain of G is a fundamental domain.*

Proof. The cardinality of an i-fundamental domains is monotone increasing with i by the definitions. Since this size is bounded above by twice the degree of G, a maximum is achieved for some n. $\square$

From now on we fix a fundamental domain Δ of G. We define a map $\phi : \mathcal{F}(G) \to \Delta$ by letting $\phi(f)$ be the unique flag in Δ in the orbit of f under $Aut(G)$; the existence and uniqueness of such a flag follow from the transitivity of G and the definition of a fundamental domain. By the *colour* of a flag f we will mean the flag $\phi(f)$ of Δ.

Our next observation is that a similar map can be defined on the flags of any r-locally-G graph for r at least as large as the n of Lemma 3.1:

Lemma 3.2. *Let $n \in \mathbb{N}$ be such that every n-fundamental domain of G is a fundamental domain, and let H be an n-locally-G graph. Then for every $x \in V(H)$, and every two isomophisms $\pi, \pi' : D_n(x; H) \to D_n(o; G)$, the compositions $\phi\pi, \phi\pi'$ coincide.*

Proof. Suppose, to the contrary, that $\phi\pi(f) \neq \phi\pi'(f)$ for some $f \in \mathcal{F}(H)$. Then letting $g := \pi(f)$, we have $\phi\pi\pi^{-1}(g) \neq \phi(g)$. But as $\pi\pi^{-1} \in Aut(D_n(o; G))$, this contradicts the fact that Δ is an n-fundamental domain of G, which holds by Lemma 3.1. $\square$

This allows us to define a map $\phi_H : \mathcal{F}(H) \to \Delta$ by letting $\phi_H(f)$ be the unique flag in Δ that equals $\phi\pi(f)$ for some isomophism $\pi : D_n(x; H) \to D_n(o; G)$. Again, the *colour* of a flag h of H is $\phi_H(h) \in \mathcal{F}(\Delta)$.

We let $r := n + 1$ for the rest of this section.

Lemma 3.3. *Let H be an r-locally-G graph, let $v \in V(G), x \in V(H)$. Let f be a flag of v in G and h a flag of x in H such that $\phi(f) = \phi_H(h)$. Then there is a unique isomorphism i from $D_r(v; G)$ to $D_r(x; H)$ such that $i(f) = h$.*

Proof. Let $\pi : D_r(o; G) \to D_r(x; H)$ be an isomophism, which exists by the definition of r-locally-G. Then $\phi\pi(h) = \phi(f)$ by Lemma 3.2.

By the definition of Δ and Lemma 3.1, there is an automorphism a of G mapping v to o with $a(f) \in \Delta$. Let $a' : D_r(v) \to D_r(o)$ be the restriction of a to $D_r(v)$. Then the composition $\pi a'$ is the desired isomorphism from $D_r(v; G)$ to $D_r(x; H)$. $\square$

Lemma 3.4. *Let H be an r-locally-G graph, and let c be an isomorphism from a face-boundary F of G to a a face-boundary of H (recall Definition 2.6). Suppose that for some flag f of F, we have $\phi(f) = \phi_H(c(f))$. Then for every flag f' of F, we have $\phi(f') = \phi_H(c(f'))$.*

Proof. By Lemma 3.3, there is an isomorphism i from $D_r(v)$ to $D_r(c(v))$ with $i(f) = c(f)$. In particular, $i(F) = c(F)$ and i extends c. Given any $x \in V(F)$, let i' denote the restriction of i to $D_n(x)$, recalling that $n = r - 1$ and n satisfies the condition of Lemma 3.2. As x and v lie on a common face F, we have $D_n(x) \subseteq D_r(v)$ and so i' is an isomorphism from $D_n(x)$ to $D_n(i(x))$. By

Lemma 3.2 the colour of any flag $g = \{x, e, F\}$ coincides with the colour of $i'(g)$. As $i'(g) = i(g) = c(g)$ (recall i extends c), our claim follows.

□

We can now prove our main result, which strengthens Theorem 1.1.

Lemma 3.5. *Let H be an r-locally-G graph. Let $f = \{v, e, F\}, h = \{x, e', F'\}$ be flags of G, H respectively, such that $\phi(f) = \phi_H(h)$. Then there is a unique cover c from G to H such that $c(f) = h$. This cover is normal.*

Proof. We are going to construct the cover c inductively, starting with the face F of f and then mapping the surrounding faces one by one.

The first step is straightforward: we set $c_0(v) = x$, and let c_0 map the remaining vertices of F to F' in the right order, so that $c_0(f) = h$. We remark that c_0 preserves colours of flags by Lemma 3.4 since it does so for f.

For the inductive step, we let C_0 be the cycle bounding F, and for $i = 1, 2, \ldots$ we assume that C_{i-1} is a cycle in G and that we have already defined a map c_{i-1} from the intersection of G with the inside of C to H in such a way that the following conditions are all satisfied:

(i) c_{i-1} preserves colours;

(ii) the restriction of c_{i-1} to $E(v)$ is injective for every $v \in V(G)$, and if $e, e' \in E(v)$ lie in a common face-boundary, then so do $c_{i-1}(e), c_{i-1}(e')$ (in other words, c_{i-1} preserves the cyclic ordering of the edges around any vertex); and

(iii) for every edge e in the domain of c_{i-1} (by which we mean that both endvertices of e are in the domain), some face-boundary of G containing e is mapped by c_{i-1} injectively to a face-boundary of H.

We are going to obtain the cycle C_i from C_{i-1} by attaching an incident face-boundary F_i. To make sure that every face is mapped at some point, we can fix an enumeration $(D_n)_{n \in \mathbb{N}}$ of the face-boundaries of G. Then, at step i we consider the minimum n such that D_n shares one or more edges with C_{i-1} but does not lie inside C_{i-1}, and moreover, $D_n \cap C_{i-1}$ is a path, and let F_i be this D_n. To see that F_i is well-defined, note that if some D_j satisfies all above requirements except the last one then, $D_j \cup C_{i-1}$ bounds a region A of $\mathbb{R}^2$ containing only finitely many faces; this is true because every 1-ended planar graph admits an embedding in the plane without accummulation points of vertices [10]. Each one of these faces D is a candidate for F_i, and for those that also fail the requirement that $D \cap C_{i-1}$ is a path, there is a corresponding region A_D strictly contained in A. As there are only finitely many such candidates, it is easy to see that at least one of them satisfies all above requirements, and we can choose it as F_i. This argument also easily implies that each D_j will be chosen as F_i at some step i.

Since $F_i \cap C_{i-1}$ is a path, and it contains an edge, $F_i \triangle C_{i-1}$ is a cycle, which we declare to be C_i. It remains to extend c_{i-1} to c_i by mapping $F_i \backslash C_{i-1}$ to H in a way that preserves flag colours.

Let w be an end-vertex of the path $P := F_i \cap C_{i-1}$ We claim that there is a unique face-boundary B of H incident with $c_{i-1}(w)$ such that (I) $c_{i-1}(P) \subseteq B$,

(II) there is an edge of $E(c_{i-1}(w)) \cap B$ not in $c_{i-1}(E(w))$, and (III) $|B| = |F_i|$. To prove this, we will make use of the following observation

$$c_{i-1} \text{ maps every facial walk } W \text{ of length 3 in its domain to a facial walk.} \quad (1)$$

Indeed, let d, m, g be the three edges appearing in W in that order, and let u, v be the endvertices of m incident with d, g respectively. Let d', g' be the other two edges that lie in a common face-boundary with m and are incident with u, v respectively. Note that m is the middle edge of exactly two facial walks of length 3, namely W and $d'mg'$.

Recall that c_{i-1} preserves adjacency of edges by (ii), hence the images of d, d', g, g' participate in the 2 facial walks of length 3 in H having $c_{i-1}(m)$ as the middle edge. By (iii) we know that at least one of the walks $c_{i-1}(d)c_{i-1}(m)c_{i-1}(g)$ and $c_{i-1}(d')c_{i-1}(m)c_{i-1}(g')$ is mapped to a facial walk, and hence, if d', g' are also in the domain of c_{i-1}, so is the other by the last remark. This proves (1).

From (1) we can deduce that c_{i-1} maps every facial walk $W = e_1e_2 \dots e_k$, no matter how long, to a facial walk. Indeed, any pair of consequtive edges e_ie_{i+1} in W uniquely determines a face K_i of H containing $c_{i-1}(e_i)c_{i-1}(e_{i+1})$ by (ii) and the fact that e_ie_{i+1} is facial in G. But by (1), $K_i = K_{i+1}$ for every relevant i because $c_{i-1}(e_i)c_{i-1}(e_{i+1})c_{i-1}(e_{i+2})$ is facial.

Applying this to our path P, we deduce that $c_{i-1}(P)$ is facial, and we choose B to be the face-boundary it belongs to and, if there is a choice (which only occurs when P is a single edge), contains an edge not in $c_{i-1}(w)$. This automatically satisfies (I) and (II).

To see that (III) is also satisfied, consider the flag $g := \{w, e, F_i\}$, where e is the edge of w contained in $F_i \cap C_{i-1}$. This flag is incident with the other flag $g' := \{w, e, D\}$ containing w and e, where D lies inside C_{i-1} and is therefore in the domain of c_{i-1}. Let j denote the flag $c_{i-1}(g')$, and note that j is incident with $c_i(g)$ along the edge $c_{i-1}(e)$. Now as c_{i-1} preserves colours by (i), and the colour of any flag is uniquely determined by the colour of any of its incident flags by our definition of colour, this implies that the colour of g coincides with the colour of the flag $\{c_{i-1}(w), c_{i-1}(e), B\}$ of H. In particular, we have $|B| = |F_i|$ by our definition of colour. This completes the proof of our claim.

We now obtain c_i by extending c_{i-1} in such a way that $c_i(F_i) = B$. Note that there is a unique such extension as c_{i-1} already maps a non-trivial subpath of F_i to B.

By our last remark, c_i preserves the colour of the flag g. By Lemma 3.4, c_i preserves the colours of all flags of F_i, and as it extends c_{i-1}, our inductive hypothesis (i) that all flag colours are preserved is satisfied. Condition (II) in the choice of B ensures that (ii) is also satisfied. Finally, (iii) is satisfied by the construction of c_i.

Thus our inductive hypothesis is preserved. Letting $c := \bigcup_i c_i$ we obtain a map from $V(G)$ to $V(H)$, which is a cover by (ii).

To see that c is unique, note that c_0 was uniquely determined by f, h, and at each step i, the map c_i was the unique way to extend c_{i-1} while keeping it a candidate for being the restriction of a cover because B was uniquely determined by F_i and c_{i-1}.

This uniqueness combined with the definition of Δ easily implies that c is normal. Indeed, Suppose $c(v) = c(w) = x$ for $v, w \in V(G)$. Let h be a flag of

x, and let f_v, f_w be the flags of v, w respectively such that $c(f_v) = h = c(f_w)$. Then $\phi(f_v) = \phi_H(h) = \phi(f_w)$. Therefore, there is an automorphism α of G such that $\alpha(v) = w$ and $\alpha(f_v) = f_w$. Note that $c \circ \alpha$ is a cover of H by G, and that $(c \circ \alpha)(f_v) = c(f_w) = h$. But as $c(f_v) = h$ too, the uniqueness of c proved above implies $c \circ \alpha = c$ as desired. □

Note that if G is a planar 1-ended vertex transitive graph, and f, h are flags of G with $\phi(f) = \phi(h)$, then there is an automorphism a of G with $a(f) = h$ by the definition of ϕ. Lemma 3.5, applied with $H = G$, implies that this automorphism is unique.

4 Further remarks

One could try to strengthen Problem 1.1 by demanding that there is a cover arising by taking a group-theoretic quotient of G, i.e. by imposing some further relations to G, so that the covered graph is a Cayley graph of a quotient of the group of G. However, the following example shows that this is not possible even in the abelian case: we construct a Cayley graph G and a family $K = K(l, k)$ of $l-1$-locally-G graphs such that G covers K but K is not even vertex-transitive.

Example: Let G' be the Cayley graph of $Z \times Z/k$ with the standard generators $(1, 0), (0, 1)$. Let G be the graph obtained from the union of two disjoint copies of G' after joining every vertex x of the first copy to every vertex in the second copy that is at the same 'height', i.e. has the same first coordinate (and so x obtains k new neighbours in the other copy). Easily, G is a Cayley graph.

Let H be a toroidal grid of 'length' l and 'width' k (you may fix k to 4, say). We index the vertices of H as $x_i^j, 0 \leq i < l, 0 \leq j < k$, so that the neighbours of any x_i^j are $x_{i-1}^j, x_{i+1}^j, x_i^{j-1}, x_i^{j+1}$, where all lower indices are $\mod l$ and upper ones $\mod k$. Let H' be a copy of H with its vertices indexed by y_i^j as above. Modify H' into a new graph H'' by rerouting one level of its edges: for every $0 \leq j < k$, we remove the edge from x_0^j to x_1^j and add an edge from x_0^j to x_1^{j+1}. Note that both H and H'' look locally like a toroidal grid when $k >> l$, but $H \cup H''$ is not vertex transitive: it has no automorphism mapping H to H''.

Let us now add some edges to $H \cup H''$ to make it connected: for every i, we join each of the k vertices in $\{x_i^j \mid 0 \leq j < k\}$ to each of the k vertices in $\{x_i^j \mid 0 \leq j < k\}$ by a new edge, and let K be the resulting graph.

Note that for every $v, w \in V(K)$, the balls of radius $diam(K) - 1 - \left\lfloor \frac{l}{2} \right\rfloor$ around v and w are isomorphic, yet K is not vertex transitive. Still, it is easy to see that G covers every such K.

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